Also solved by L. A. H. WARREN, W. C. EELLS, GEORGE PAASWELL, E. H. WORTHINGTON, HORACE OLSON, G. L. WAGAR, H. N. CARLETON, E. J. OGLESBY, H. C. FEEMSTER, BENJ. SINITSSCY, W. J. THOME, C. C. YEN, R. M. MATHEWS, E. F. CANADY, N. P. PANDYA, J. J. GINSBURG, O. S. ADAMS, A. M. HARDING, and PAUL CAPRON.

GEOMETRY.

485. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find the surface generated by the orthogonal projection of a given line upon a variable plane turning about a fixed axis.

I. SOLUTION BY ELIJAH SWIFT, University of Vermont.

Take the fixed axis as X-axis, and the common perpendicular of this line and the given line as Z-axis. Call the angle θ made by the line we are projecting and the X-axis, so that the given line has the direction cosines $\cos\theta$, $\sin\theta$, 0; let the coördinates of the point where this line intersects the Z-axis be $(0, 0, a)$. Then the equation of the variable plane may be written in the form

(A) $y - \lambda z = 0$, where λ is a parameter.

The equation of any plane through the given line may be written in the form

(B) $\sin\theta x - \cos\theta y + k(z - a) = 0$, where k is an arbitrary constant.

If (B) is perpendicular to (A), k must equal $-\cos\theta/\lambda$. Substituting this value for k in (B), (A) and (B) give the required equation in parameter form. Eliminating λ, we obtain the explicit equation

$$\tan\theta \cdot xy = y^2 + z^2 - az.$$

Rotating the axes through an angle $\theta/2$ about the Z-axis, and changing the origin to the point $(0, 0, a/2)$, the equation finally becomes

$$x^2\left(\frac{1 - \cos\theta}{2\cos\theta}\right) - y^2\left(\frac{1 + \cos\theta}{2\cos\theta}\right) - z^2 = -\frac{a^2}{4},$$

which is the equation of a hyperboloid of one sheet.

II. SOLUTION BY THE PROPOSER.

The orthogonal projection u of a given line p upon a plane β passing through a given axis q, may be obtained as the intersection of β with the plane α perpendicular to β and passing through p. When the plane β turns about q, the projecting plane α turns about p; to each position of β corresponds one and only one position of α, and vice versa. The line u is, therefore, the intersection of two corresponding planes α, β of two projective pencils (p), (q).

I. The lines p, q are coplanar. *The line u generates a cone of second degree (C), of which p, q are elements.*

The tangent planes to (C) along p and q are the planes perpendicular to the plane (pq) and passing through the lines p and q respectively. The line of intersection of these tangent planes is perpendicular to (pq) and is the polar ray of this plane with respect to (C). Hence: *The plane of the two given lines is a plane of symmetry of the cone.*

The orthogonal projection P' of any point P of p upon the plane β lies on the line u and in the plane π through P perpendicular to q. When β varies, P' describes, in the plane π, a circle having for diameter the segment joining P to the point of intersection of π with q. This circle is the curve of intersection of (C) with the plane π. Similarly for the orthogonal projection Q' of any point Q of q upon α. Hence: *The planes perpendicular to the given lines are the planes of the circular sections of the cone.*

This cone is sometimes called the *Orthogonal* cone (Theodor Reye, *Geometrie der Lage*, part I, p. 119, fifth edition).

If the given lines p, q are parallel, the cone becomes a cylinder of revolution.

II. The lines p, q are skew. *The line u generates a hyperboloid of one sheet (H), of which p, q are two rays of the same system.*

The planes perpendicular to the given lines are the planes of circular sections of the hyperboloid. Same proof as for the cone above.

If the two pencils of planes (p), (q) be transported parallel to themselves so as to have their axes p, q pass through the center of (H), they will generate the asymptotic cone of the surface. Hence: *The asymptotic cone of the hyperboloid is orthogonal.*

This hyperboloid is sometimes called the *Orthogonal* hyperboloid (Salmon, *Analytic Geometry of Three Dimensions*, Vol. I, p. 116, fifth edition).

REMARK.—From the method of construction of the line u it follows that the cone (C) (or the hyperboloid) is also the surface generated by the orthogonal projection of the line q upon a variable plane turning about the axis p.

The reader may consider the special cases when (1) the given lines are perpendicular to each other; (2) one or both of the given lines are at infinity.

Also solved by H. C. FEEMSTER, G. W. HARTWELL, and FRANK H. LOUD.

486. Proposed by ARON INGVALE, Brooklyn, N. Y.

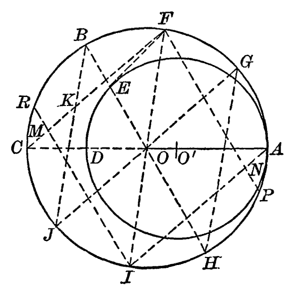
Does the following construction trisect an angle? With the vertex, O, of the given angle as center and with a radius R, describe a circle intersecting the sides of the given angle in A and B. With a radius $\frac{3}{4}R$, and center O' on OA, describe a circle tangent to the other circle at A and cutting the other side of the angle at E. At E draw a tangent to the last circle and produce it to meet the first circle at F. Draw FO. Then is angle BOF one-third of the angle BOA?

REMARK. Though the construction does not, of course, lead to the trisection of an angle in general, yet as a first approximation it is very good. This fact together with the fact that the construction is very simple, and that the proposer's demonstration that it does trisect the angle is very illusive, are the reasons for giving the problem a place in the MONTHLY.

EDITORS.

SOLUTION BY THE PROPOSER.

Let OG bisect $\angle AOF$. Extend FO and GO till they meet the original circle at I and J respectively. Also extend BO to H. Draw FC and AI, which for obvious reasons are parallel to each other and to GJ.



Then draw FP and IR each parallel to BH and prove $\angle BJG = \angle FIA$ as follows:

$\triangle BOJ$ is isosceles and since $CF \parallel JG$, therefore $\triangle KEB$ is also isosceles. But $\triangle KEB$ is similar to $\triangle FNI$ because their sides are respectively parallel.

Hence $\triangle FNI$ is isosceles and also $\triangle IMF$ to which it is congruent.

Hence $\angle CFI = \angle IFP = \angle BJG = \angle FIA$, from which follows $\angle BOF = \angle GOF$. But since $\angle GOF$ was constructed equal to $\angle AOG$, it follows that $\angle AOB$ is trisected.

Note.—The fallacy in this proof has been pointed out to Mr. Ingvale and he admits his error, but it is left as an exercise to others to show that the method applies to a right angle or a straight angle but fails in general. EDITORS.

CALCULUS.

403. Proposed by C. N. SCHMALL, New York City.

A paraboloid of revolution generated by the curve $x^2 = 4ay$, contains a quantity of water such that if a sphere of radius r be dropped to the bottom, it will just be covered by the water. Show that if the volume of water used in this experiment is to be a *minimum*, then we must have $a = r/6$.